Theoretical Models of CSR in Storage Rings

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1D CSR Wakefield and Impedance in Free Space

Wakefield due to CSR was given by Derbenev, Rossbach, Saldin, Shiltsev (1995) and Murphy, Krinsky, Gluckstern (1997),

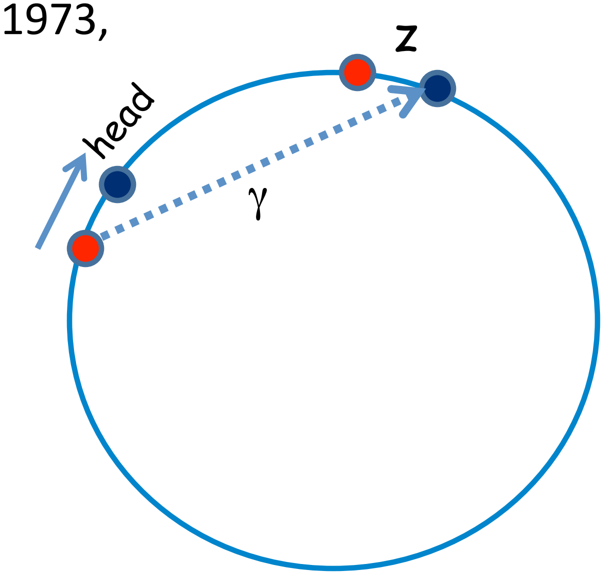
$$W(z) = \frac{4\pi\rho^{1/3}}{3^{1/3}} \frac{\partial}{\partial z} z^{-1/3}$$

For z>0. It vanishes when z<0 (force is acting on the electron ahead).

Impedance was derived by Faltens and Laslett in 1973,

$$Z_{CSR}(k) = \left(\frac{4\pi}{c}\right) \frac{\Gamma(\frac{2}{3})(\frac{\sqrt{3}}{2} + \frac{i}{2})}{3^{1/3}} (\rho k)^{1/3}$$

where ρ is the bending radius.



Its simplicity and universal

1D Wakefield of CSR with Shielding

Murphy, Krinsky, Gluckstern (1997)

For an electron moving on a circular orbit with bending radius ρ in the middle of two parallel plates that are separated by 2h, the longitudinal wakefield is given by

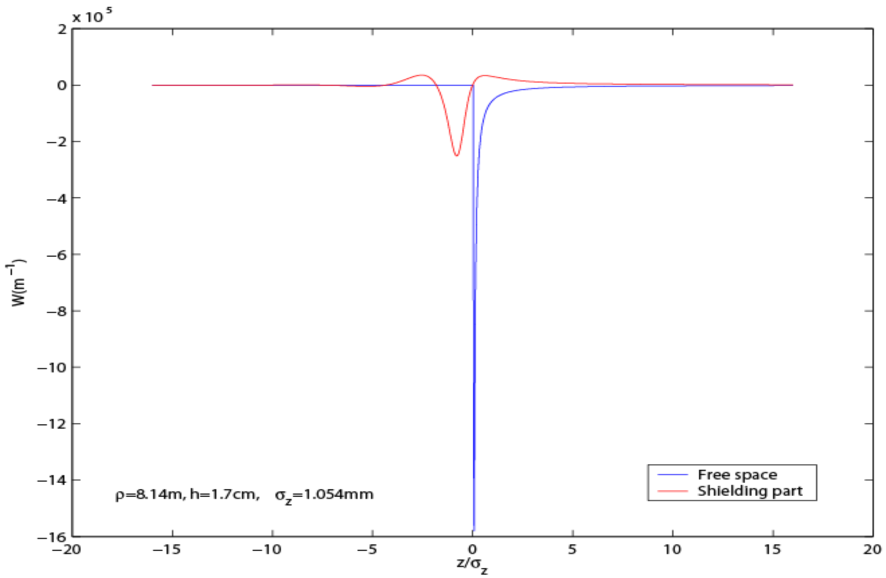
$$W(\xi) = \frac{8\pi\gamma^4}{3\rho}[w(3\gamma^3\xi) - \frac{3}{8}\frac{1}{\Delta^2\gamma^4}G_2(\xi/\Delta^{3/2})]$$

where $\xi = z/2\rho$, $\Delta = h/\rho$. The formula is valid for $|\xi| <= \Delta << 1$ and $\gamma^2\Delta >> 1$. G_2 can be written as,

$$G_2(\zeta) = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}[\frac{4Y_k^4(3-Y_k^4)}{(1+Y_k^4)^3}]$$

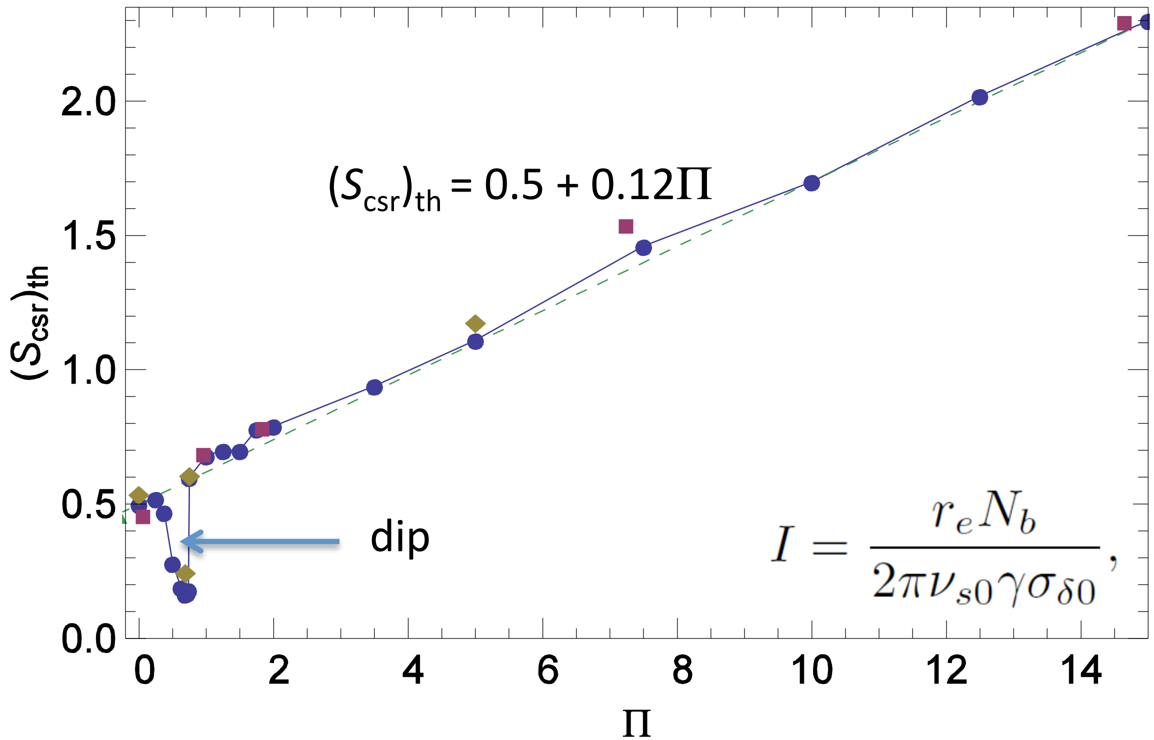
and

$$Y_k^4 - \frac{\zeta}{k^{3/2}}Y_k - 3 = 0$$



Scaling Property of Microwave Instability

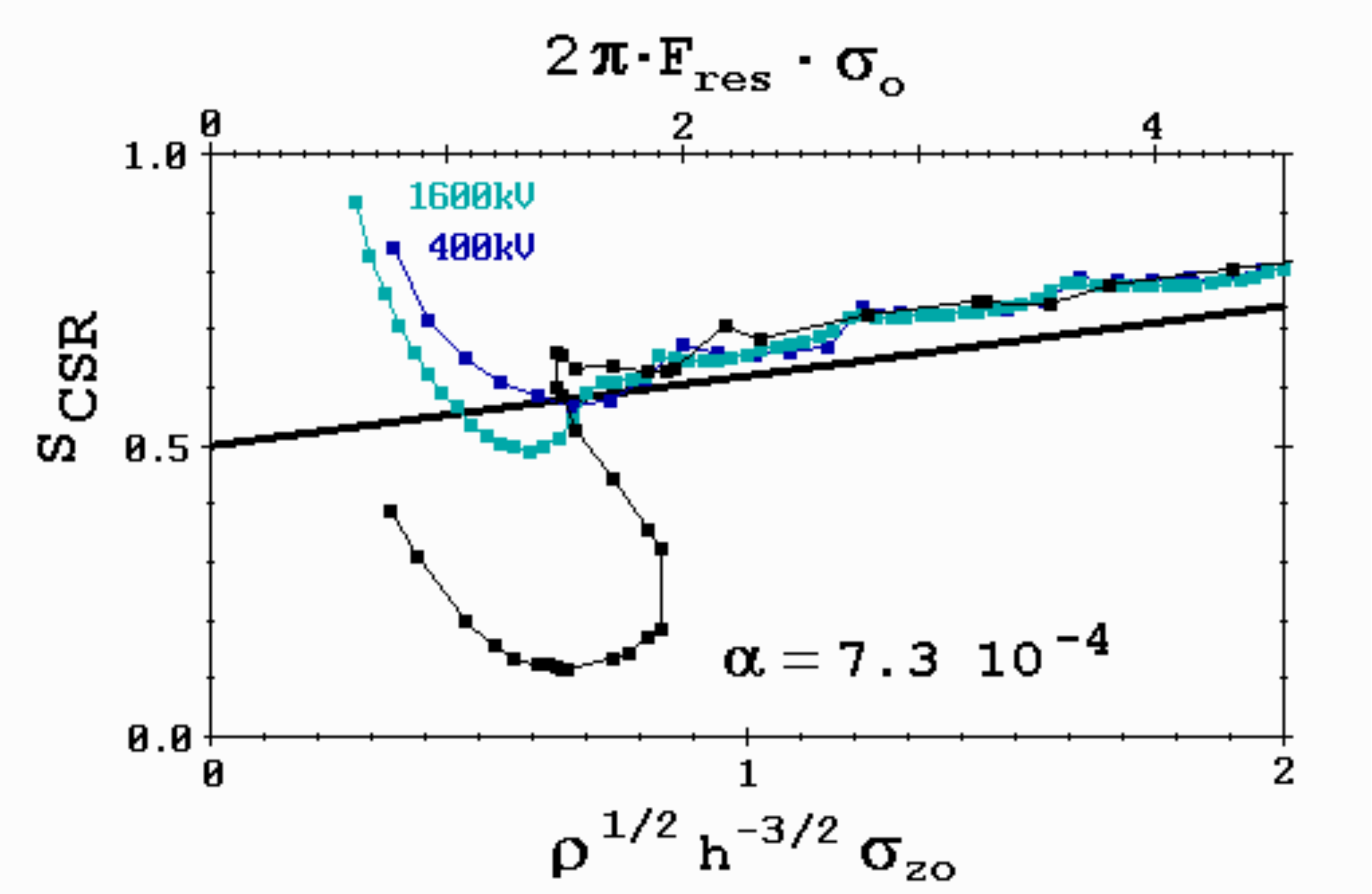
Bane, Cai, Stupakov (2010)



It can be shown that there are only two free parameters: the shielding $\Pi=\rho^{1/2}\sigma_{z0}/h^{3/2}$ and the strength $S_{csr}= I\rho^{1/3}/\sigma_{z0}^{4/3}$. *For the CSR wake, threshold value of S_{csr} vs. shielding parameter, Π. Symbols give results of the VFP solver (blue circles), the LV code (red squares), and the VFP solver with twice stronger radiation damping (olive diamonds).*

Shielded CSR-Wake – BESSY II

$$S_{csr} = \frac{N r_e}{2\pi \nu_s \gamma \sigma_\varepsilon} \cdot \rho^{1/3} (c\sigma_0)^{-4/3} \qquad F_{res} = c\sqrt{\pi / 24}\,\rho^{1/2} h^{-3/2}$$



Courtesy of Peter Kuske (Talk at SLAC 2017)

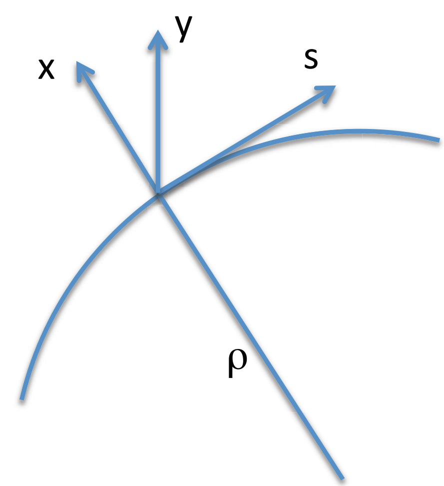
Transverse Force in Curved Geometry

Equation of motion:

$$x'' + \frac{x}{\rho^2} = \frac{\delta}{\rho} + \frac{e}{c p_0 \beta_s}[E_x + \beta_y B_s - \beta_s (1+\frac{x}{\rho}) B_y],$$

$$y'' = \frac{e}{c p_0 \beta_s}[E_y + \beta_s (1+\frac{x}{\rho}) B_x - \beta_x B_s]$$

Curvature terms



The curvilinear coordinate

- Curvature terms are conceptually important
- E_x, E_y, B_x, B_y, and B_s are the self-fields
- No explicit dependence on the potentials
- Equations are derived from the Hamiltonian by Courant-Synder

Lienard-Wiechert Formula

Space Charge

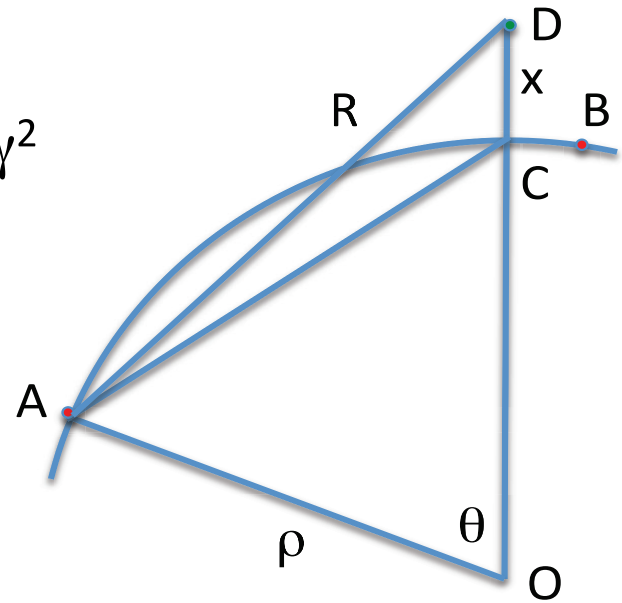
Radiated Field

$$\vec{E} = e\left[\frac{\vec{n}-\vec{\beta}}{\gamma^2(1-\vec{n}\cdot\vec{\beta})^3 R^2}\right]_{ret} + \left(\frac{e}{c}\right)\left[\frac{\vec{n}\times(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}}{(1-\vec{n}\cdot\vec{\beta})^3 R}\right]_{ret},$$

$$\vec{B} = \vec{n}\times\vec{E}$$

- Space charge is suppressed by $1/\gamma^2$
- Identify radiated field with CSR
- Subject to retarded condition:

$$t' = t - \frac{R}{c}$$



Electrical and Magnetic Fields

$$E_s = \frac{e\beta^2[\cos 2\alpha - (1+\chi)][(1+\chi)\sin 2\alpha - \beta\kappa]}{\rho^2[\kappa - \beta(1+\chi)\sin 2\alpha]^3}$$

$$E_x = \frac{e\beta^2 \sin 2\alpha[(1+\chi)\sin 2\alpha - \beta\kappa]}{\rho^2[\kappa - \beta(1+\chi)\sin 2\alpha]^3}$$

$$B_y = \frac{e\beta^2\kappa[(1+\chi)\sin 2\alpha - \beta\kappa]}{\rho^2[\kappa - \beta(1+\chi)\sin 2\alpha]^3}$$

where

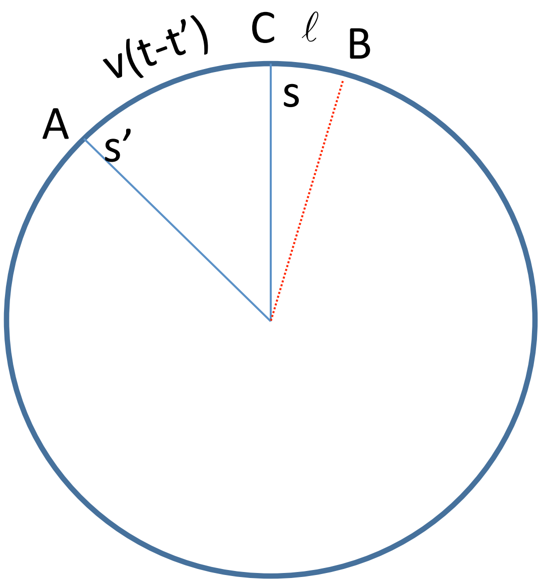
$$\kappa = \frac{R}{\rho} = \sqrt{\chi^2 + 4(1+\chi)\sin^2\alpha},$$

$$\alpha = \theta / 2,$$

$$\chi = x / \rho$$

- They are simplest expressions, especially in the denominator and chosen to suppress the numerical noise near the singularity.

Retarded Time and Longitudinal Position



Retarded Time:

$$t' = t - \frac{R}{c}$$

Time of flight at position s:

$$\ell = v(t - t') - (s - s')$$

It is the variable for the wake. The arc distance to the source particle at the time t. We derive its relation to α.

$$\xi = \alpha - \frac{\beta}{2}\sqrt{\chi^2 + 4(1+\chi)\sin^2\alpha}$$

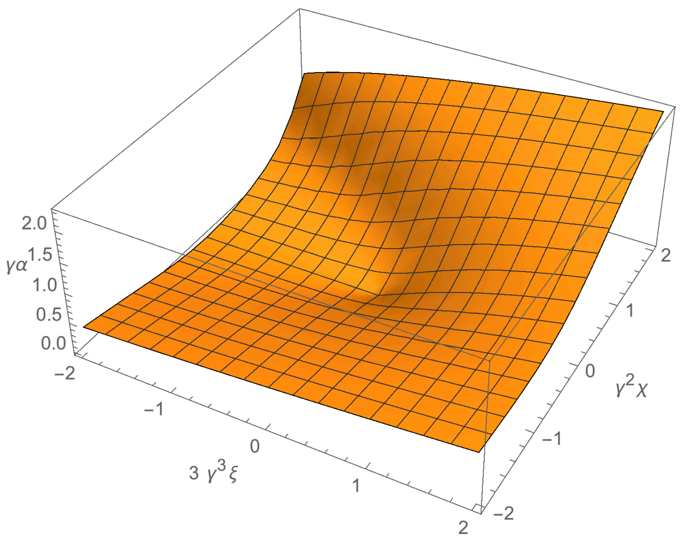
where $\xi = -\ell/2\rho$ and $\ell = z' - z$.

Solutions of the Retarded Condition

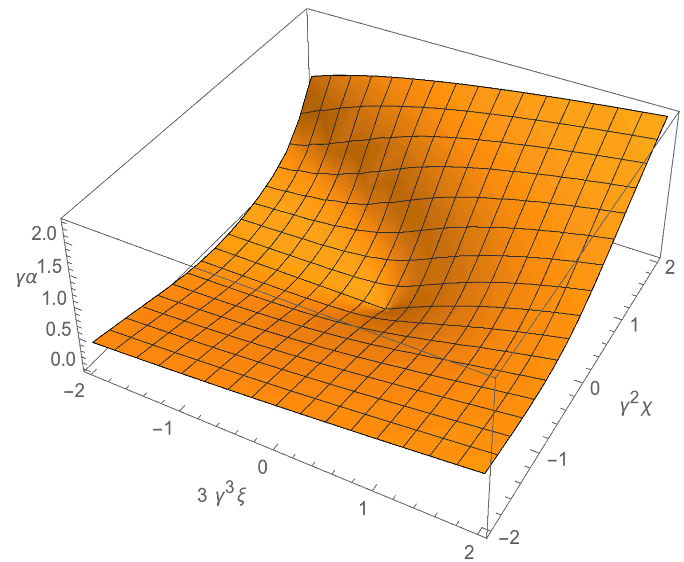
Expanding up to the fourth-order of α of the retarded condition, we have

$$\alpha^4 + \frac{3(1-\beta^2-\beta^2\chi)}{\beta^2(1+\chi)}\alpha^2 - \frac{6\xi}{\beta^2(1+\chi)}\alpha + \frac{3(4\xi^2-\beta^2\chi^2)}{4\beta^2(1+\chi)} = 0$$

Numerical



Analytical



Numerical solution is on mesh: 512x512 using Mathematica taking several hours.
The differences between the numeric and analytic solutions are at an order of 10^{-6}.
Here we have used γ=500.

Analytical Solution of the Retarded Condition

In general, we want to find the roots of the depressed quartic equation:

$$\alpha^4 + \upsilon\alpha^2 + \eta\alpha + \zeta = 0$$

It has analytical solution discovered by Ferrari (1522-1565) by adding and subtracting a term to make a difference of two perfect squares. To find the term, we need to first find the roots of a third-order equation. A root m is given by,

$$m = -\frac{\upsilon}{3} + \left(\frac{\zeta}{3} + \frac{\upsilon^2}{36}\right)\Omega^{-1/3} + \Omega^{1/3}$$

where

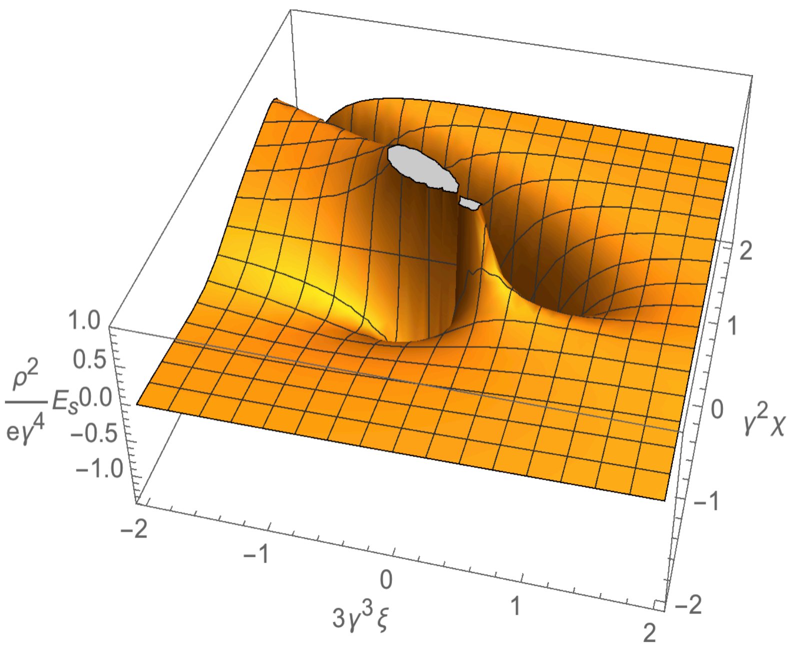
$$\Omega = \frac{\eta^2}{16} - \frac{\zeta\upsilon}{6} + \frac{\upsilon^3}{216} + \sqrt{\left(\frac{\eta^2}{16} - \frac{\zeta\upsilon}{6} + \frac{\upsilon^3}{216}\right)^2 - \left(\frac{\zeta}{3} + \frac{\upsilon^2}{36}\right)^3}$$

The solution of α:

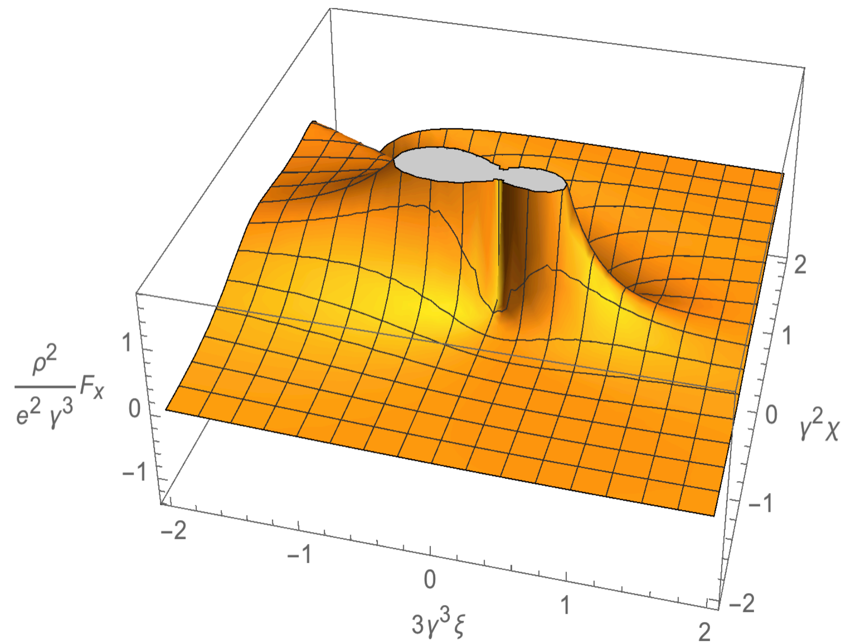
$$\alpha = \begin{cases} \frac{1}{2}\left(\sqrt{2m} + \sqrt{-2(m+\upsilon) - \frac{2\eta}{\sqrt{2m}}}\right) & \xi >= 0 \\ \frac{1}{2}\left(-\sqrt{2m} + \sqrt{-2(m+\upsilon) + \frac{2\eta}{\sqrt{2m}}}\right) & \xi < 0 \end{cases}$$

Longitudinal Field and Centrifugal Force

$\rho^2 E_s / e\gamma^4$

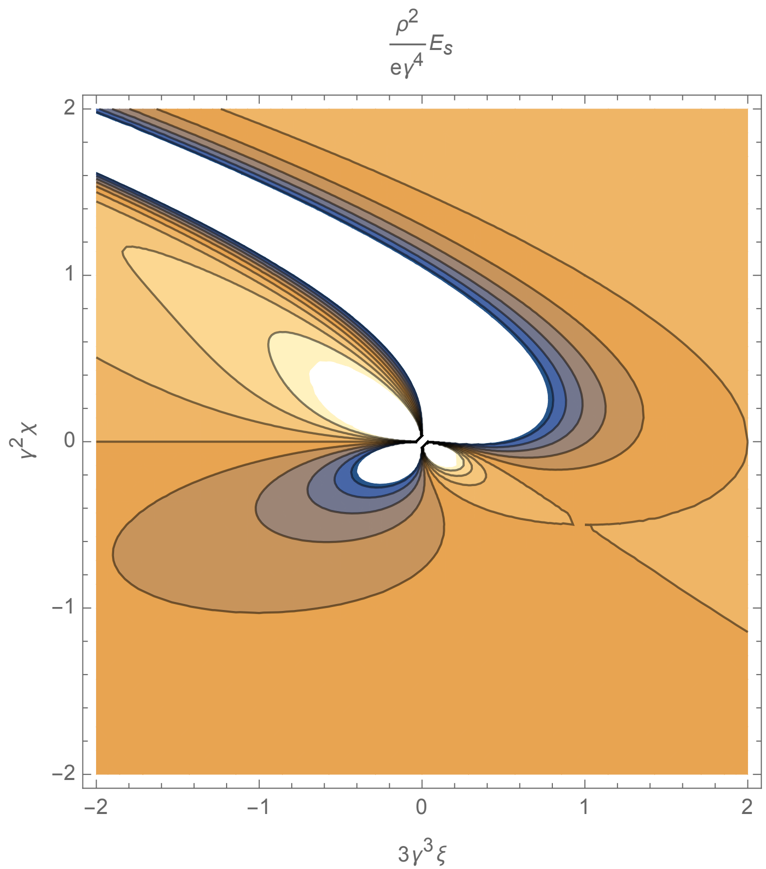


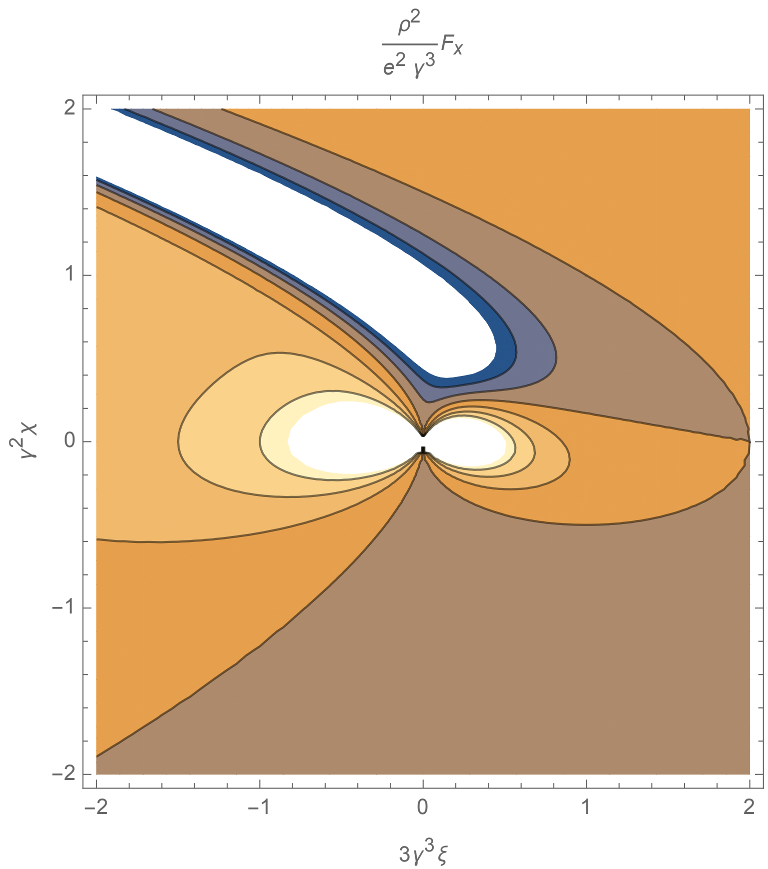
$\rho^2 F_x / e^2\gamma^3$



- The scaling with respect to γ is different. Here we have used γ=500.
- The centrifugal force is much hard to compute numerically because of the cancellation between the electric and magnetic forces.

Longitudinal Field and Centrifugal Force





- The scaling with respect to γ is different. Here we have used γ=500.
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A Longitudinal Potential Ψ_s

Differentiate the retarded condition,

$$\xi = \alpha - \frac{\beta}{2}\sqrt{\chi^2 + 4(1+\chi)\sin^2\alpha}$$

We have,

$$d\xi = \left(1 - \frac{\beta(1+\chi)\sin 2\alpha}{\sqrt{\chi^2 + 4(1+\chi)\sin^2\alpha}}\right)d\alpha$$

Combining it with the longitudinal electric field E_s, we find

$$E_s d\xi = \frac{e\beta^2[\cos 2\alpha - (1+\chi)][(1+\chi)\sin 2\alpha - \beta\kappa]}{\rho^2\kappa[\kappa - \beta(1+\chi)\sin 2\alpha]^2} d\alpha$$

$$= d\left(\frac{e\beta^2\left(\cos 2\alpha - \frac{1}{1+\chi}\right)}{2\rho^2[\kappa - \beta(1+\chi)\sin 2\alpha]}\right)$$

or

$$E_s = \frac{\partial \psi_s}{\partial \xi}$$

where

$$\psi_s(\xi,\chi) = \frac{e\beta^2\left(\cos 2\alpha - \frac{1}{1+\chi}\right)}{2\rho^2[\kappa - \beta(1+\chi)\sin 2\alpha]}$$

Transverse Force and Potential Ψ_x

Similarly,

$$\psi_x(\xi,\chi)=\frac{e^2\beta^2}{2\rho^2}\{\frac{1}{|\chi|(1+\chi)}[(2+2\chi+\chi^2)F(\alpha,\frac{-4(1+\chi)}{\chi^2})-\chi^2E(\alpha,\frac{-4(1+\chi)}{\chi^2})]$$

$$+\frac{\kappa^2-2\beta^2(1+\chi)^2+\beta^2(1+\chi)(2+2\chi+\chi^2)\cos 2\alpha-\kappa\beta(1+\chi)\sin 2\alpha[1-\beta^2(1+\chi)\cos 2\alpha]}{[\kappa^2-\beta^2(1+\chi)^2\sin^2 2\alpha]},$$

where, $F_x=\frac{\partial\psi_x}{\partial\xi}$

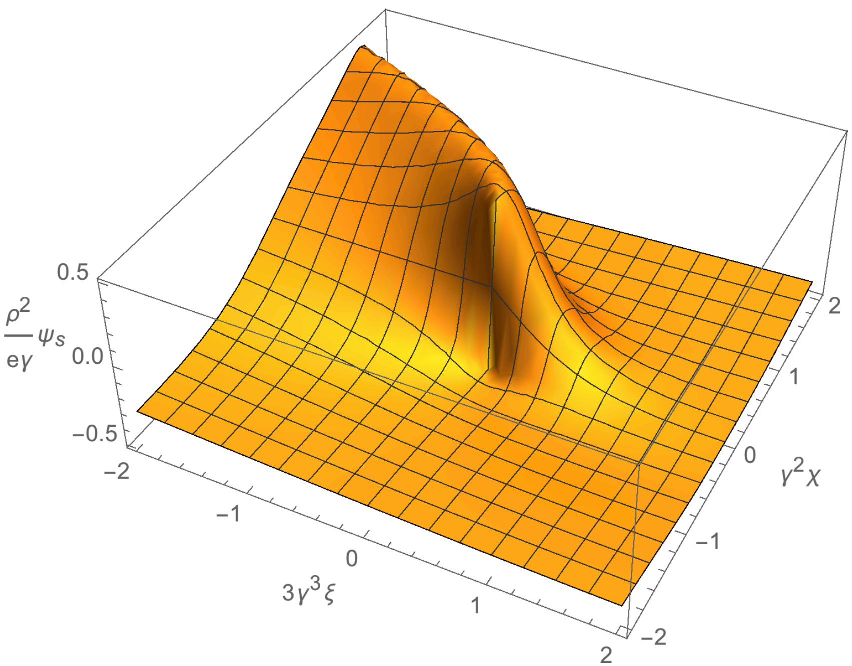
Curvature term

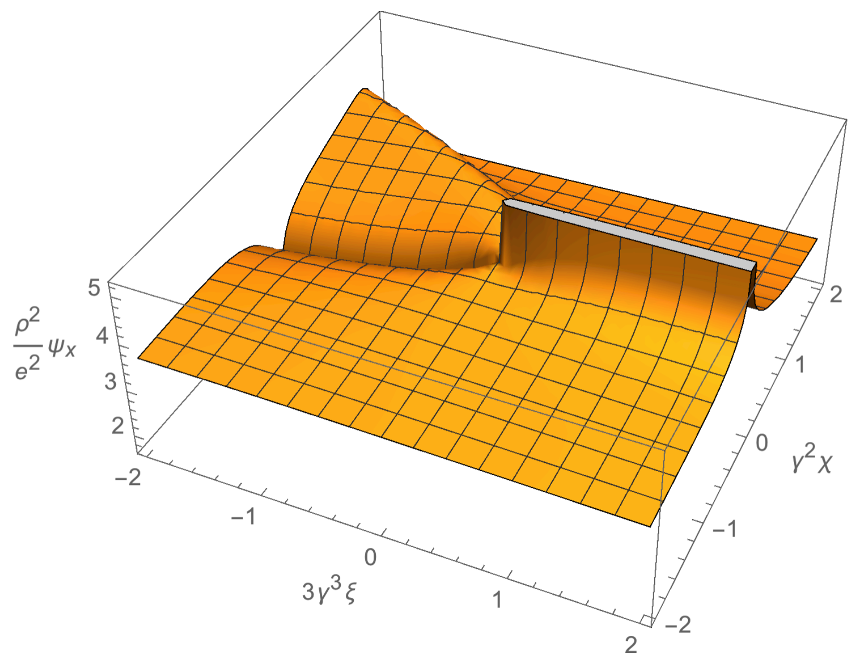
and, $$F_x=\frac{e\beta^2[\sin 2\alpha-(1+\chi)\beta\kappa][(1+\chi)\sin 2\alpha-\beta\kappa]}{\rho^2[\kappa-\beta(1+\chi)\sin 2\alpha]^3}$$

- The Transverse force is the Lorentz force and plus the curvature term
- The curvature term is necessary for the analytical expression
- $F(\alpha,k)$ and $E(\alpha,k)$ are the incomplete elliptic integrals of the first and second kind
- F_y=0, so the particles stay in the plane if they are initially in the horizontal plane

Longitudinal and Transverse Potentials

$\rho^2\Psi_s/e\gamma$ $\rho^2\Psi_x/e^2$





- The scaling with respect to γ is different. Here we have used γ=500.
- The "logarithmic" singularity is clearly seen in the transverse potential
- along the line of χ=0.

Wakefields

From the equations of the motion, the changes of the momentum deviation and kick are given by,

$$\delta' = \frac{r_e N_b}{\gamma} W_s(z,\chi),$$

$$x'' = \frac{r_e N_b}{\gamma} W_x(z,\chi)$$

where r_e is the classical electron radius, N_b the bunch population, and the wakes,

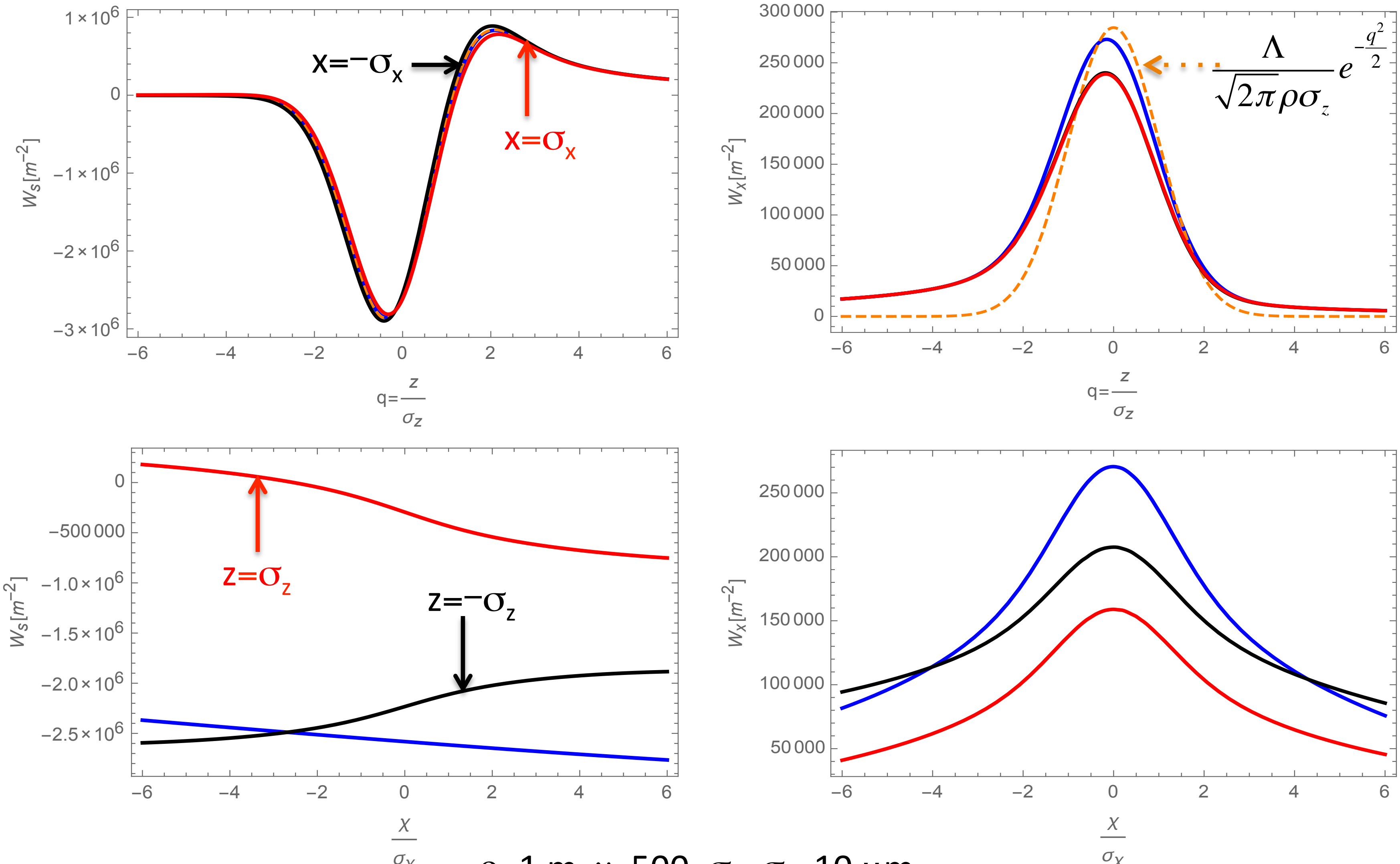
$$W_s(z,\chi) = \iint Y_s\left(\frac{z-z'}{2\rho}, \chi-\chi'\right)\frac{\partial\lambda_b(z',\chi')}{\partial z'} dz' d\chi',$$

$$W_x(z,\chi) = \iint Y_x\left(\frac{z-z'}{2\rho}, \chi-\chi'\right)\frac{\partial\lambda_b(z',\chi')}{\partial z'} dz' d\chi',$$

with $Y_s=2\rho\Psi_s/(e\beta^2)$, $Y_x=2\rho\Psi_x/(e\beta)^2$ and λ_b is the normalized distribution.

- These are additional changes when integrating through the bend.

Gaussian Bunch Wakes



ρ=1 m, γ=500, $\sigma_x=\sigma_z$=10 μm,

$$\Lambda = \frac{11}{24}\gamma_E - 4 + \ln(\frac{2\rho^2}{\sigma_x^2}) + \frac{13}{24}\ln(\frac{\sigma_z^2}{2\rho^2})$$

Estimate of Emittance Growth

Increase of the projected emittance:

$$\Delta\varepsilon_N = \frac{1}{2}\gamma\beta_x < (\Delta x' - <\Delta x'>)^2 >,$$

From the longitudinal contribution a bending magnet:

$$\Delta\varepsilon_N = 7.5\times10^{-3}\frac{\beta_x}{\gamma}\left(\frac{N_b r_e L_B^2}{\rho^{5/3}\sigma_z^{4/3}}\right)^2,$$

It leads to 38% increase of the emittance for the last dipole. From the centrifugal force, we have

$$\Delta\varepsilon_N = \frac{(-3+2\sqrt{3})}{24\pi}\frac{\beta_x}{\gamma}\left(\frac{\Lambda N_b r_e L_B}{\rho\sigma_z}\right)^2,$$

This gives 29% increase of the emittance.

The parameters for the last bend of BC2 in LCLS

Symbol	γ	ε_N	σ_z	N_b	β_x	ρ	L_B
Value	10,000	0.5 μm	10 μm	10^9	5 m	5 m	0.5 m

Conclusion

- The transverse force in the curveted coordinate is essentially the Lorentz force but with a substitution of the transverse magnetic field, $B_{x,y} \to (1+x/\rho)B_{x,y}$
- The curvature term play a key role for deriving the point-charge wakefield explicitly in terms of the incomplete elliptic integrals of the first and second kind
- Emittance growth due to the centrifugal force is at the same level of the contribution through the energy changes
- A steady-state theory of the coherent synchrotron radiation in two-dimensional free space is developed

References

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6) Ohmi's talk in theory club, SLAC 2016

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